

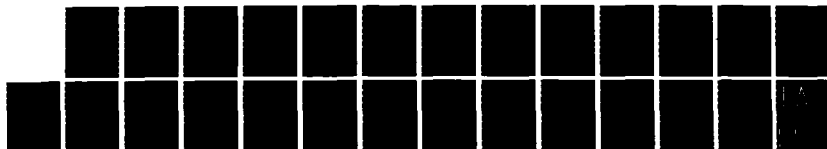
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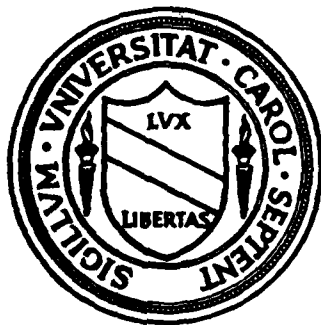
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CONTINUITY OF GAUSSIAN PROCESSES

by

Gennady Samorodnitsky

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Technical Report No. 149

August 1986

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SECURITY CLASSIFICATION OF THIS PAGE

AD-A174738

## REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED			1b. RESTRICTIVE MARKINGS		
2a. SECURITY CLASSIFICATION AUTHORITY NA			3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for Public Release; Distribution Unlimited		
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE NA					
4. PERFORMING ORGANIZATION REPORT NUMBER(S) Technical Report No. 149			5. MONITORING ORGANIZATION REPORT NUMBER(S) <b>AFOSR-TR. 86-2043</b>		
6a. NAME OF PERFORMING ORGANIZATION University of North Carolina		6b. OFFICE SYMBOL (If applicable)		7a. NAME OF MONITORING ORGANIZATION AFOSR/NM	
6c. ADDRESS (City, State and ZIP Code) Center for Stochastic Processes, Statistics Department, Phillips Hall 039-A, Chapel Hill, NC 27514			7b. ADDRESS (City, State and ZIP Code) Bldg. 410 Bolling AFB, DC 20332-6448		
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR		8b. OFFICE SYMBOL (If applicable) nm		9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER F49620 85 C 0144	
8c. ADDRESS (City, State and ZIP Code) Bldg. 410 Bolling AFB, DC			10. SOURCE OF FUNDING NOS.		
			PROGRAM ELEMENT NO. 6.1102F		PROJECT NO. 2304
					TASK NO. A5
11. TITLE (Include Security Classification) Continuity of Gaussian processes			12. PERSONAL AUTHOR(S) Samorodnitsky, G.		
13a. TYPE OF REPORT technical preprint		13b. TIME COVERED FROM 9/85 TO 9/86		14. DATE OF REPORT (Yr., Mo., Day) August 1986	
15. PAGE COUNT 22					
16. SUPPLEMENTARY NOTATION					
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)		
FIELD	GROUP	SUB. GR.	Keywords: Gaussian processes, sample continuity, local continuity, isonormal process, metric entropy.		
XXXXXX	XXXXXXXXXX	XXXX			
19. ABSTRACT (Continue on reverse if necessary and identify by block number)					
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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED <input checked="" type="checkbox"/> SAME AS RPT. <input type="checkbox"/> DTIC USERS <input type="checkbox"/>			21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED		
22a. NAME OF RESPONSIBLE INDIVIDUAL Peggy Ravitch <i>Ma. Crowley</i>			22b. TELEPHONE NUMBER <i>202</i> (Include Area Code) <i>767</i> 919-962-2307 <i>5025</i>		22c. OFFICE SYMBOL AFOSR/NM

# CONTINUITY OF GAUSSIAN PROCESSES

by

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## Abstract

We give sufficient conditions for local continuity of the isonormal process  $L$  at some point of its parameter set. Since a Gaussian process, defined on a compact parameter space, that is a.s. continuous at each point is sample continuous, our result can be applied to the problem of general sample continuity of Gaussian processes. It is shown that our sufficient conditions are strictly weaker than the classical sufficient conditions for sample continuity.

AMS 1980 Subject Classification: 60G15, 60G17, 60G60.

Keywords and Phrases: Gaussian processes, sample continuity, local continuity, isonormal process, metric entropy.

<sup>1</sup>Research supported in part by the Fund for the Promotion of Research at the Technion under Contract 190-678, the Weizman Foundation while visiting the Center for Stochastic Processes, Chapel Hill, North Carolina, and the Air Force Office of Scientific Research Grant No. F49620 85 C 0144.

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## 1. Introduction

Let  $\{X(t), t \in C\}$  be a Gaussian process with a continuous covariance function over a compact subset  $C$  of a metric space  $(S, d)$ . Such a process is called *sample-continuous* if there is a version of the process with continuous sample functions. Equivalently,  $\{X(t), t \in C\}$  is sample-continuous if it is uniformly continuous for  $t$  restricted to a countable dense subset of  $C$ . The process is said to be *sample-bounded* if it has a version with bounded sample functions.

Let  $t_0$  be a point of  $C$ . The process is said to be *continuous at  $t_0$*  if there is a version of the process with sample functions that are continuous at  $t_0$ . Equivalently, the process is continuous at  $t_0$  if 
$$P\left(\lim_{\substack{t \rightarrow t_0 \\ t \in C^*}} X_t = X_{t_0}\right) = 1,$$
  $C^*$  being a countable dense subset of  $C$ .

Let  $H$  be a real, infinite-dimensional Hilbert space. A linear map  $L$  from  $H$  into real Gaussian variables with  $ELx = 0$ ,  $ELxLy = (x, y)$  for all  $x, y \in H$  is called the *isonormal* Gaussian process on  $H$ . (As usual,  $(\cdot, \cdot)$  denotes the inner product in  $H$ ).

A modern approach to the study of sample function continuity and boundedness of Gaussian processes reduces this problem to the study of those sets  $C \subset H$  on which the isonormal  $L$  has continuous or bounded sample functions, called GC-sets and GB-sets respectively (Dudley (1967, 1973), Feldman (1971), Sudakov (1969, 1971)). This approach relates GC and GB properties to a certain measure of the size of a set  $C$  in  $H$ , called *metric entropy*.

Let  $C$  be a subset of a metric space  $(S, d)$ . Given  $\epsilon > 0$ , let  $N(C, \epsilon) \equiv N_C(\epsilon)$  be the minimal number of points  $x_1, x_2, \dots, x_n$  from  $C$  such that for any  $y \in C$  there is an  $x_i$  such that  $d(x_i, y) \leq \epsilon$ . Then  $H_C(\epsilon) = \ln N_C(\epsilon)$  is called the metric entropy of  $C$ , and the *exponent of entropy*  $r(C)$  is defined by

$$r(C) = \lim_{\varepsilon \rightarrow 0} \frac{\ln H_C(\varepsilon)}{|\ln \varepsilon|}.$$

Dudley (1973) proved that  $C \subset H$  is always a GC-set if

$$(1.1) \quad \int_0^1 H_C(x)^{1/2} dx < \infty.$$

This is an extremely sharp result, for it implies that  $C$  is a GC-set if  $r(C) < 2$ , and it is known that  $C$  cannot be a GB-set (and so not a GC-set) if  $r(C) > 2$  (Sudakov (1969)). The case  $r(C) = 2$  includes an ambiguous range, however, where  $H_C(\varepsilon)$  cannot determine whether  $C$  is GB or GC (Dudley (1973)). In particular, there are GC-sets for which the integral (1.1) diverges.

In this paper we find conditions under which the isonormal process  $L$  on a set  $C \subset H$  is a.s. continuous at some point  $x_0 \in C$ . This is closely related to the question of whether or not  $C$  is GC. Clearly, if  $C$  is GC, then the isonormal process is a.s. continuous at each point of  $C$ . Less evident is the converse statement: if the isonormal process is a.s. continuous at every point of  $C$ , then  $C$  is a GC-set. This important property of Gaussian processes was noted for Gaussian processes on  $[0,1]$  by Marcus and Shepp (1971), page 436, who give credit for the idea to R. Dudley, and can be extended as follows.

THEOREM 1.1.

*Let  $C$  be a compact subset of a metric space  $(S,d)$ , and  $X(t)$  a Gaussian process on  $C$ . If  $X(t)$  is a.s. continuous at each point of  $C$  then it is sample continuous.*

The proof of Theorem 1.1 will be given in the Appendix at the end of the paper.

In the following section we give sufficient conditions for local continuity of the isonormal process. These conditions turn out to be strictly weaker than

I am deeply indebted to Robert Adler and Stamatis Cambanis for their valuable remarks during the preparation of this paper.

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## 2. Sufficient conditions for local continuity

We start with some additional definitions and preliminary results.

For a given set  $C$  in a Hilbert space  $H$  and a given point  $x_0 \in C$ , set

$$(2.1) \quad C_{x_0}(\delta) := \{x \in C : \|x - x_0\| \leq \delta\}, \quad \delta > 0.$$

$$(2.2) \quad N_{x_0}(\delta, \varepsilon) := N(C_{x_0}(\delta), \varepsilon), \quad \delta > 0, \quad \varepsilon > 0.$$

$$(2.3) \quad N_{x_0}(\delta_1, \delta_2, \varepsilon) := N(C_{x_0}(\delta_2) \cap \overline{C_{x_0}(\delta_1)}, \varepsilon), \quad \delta_2 > \delta_1 \geq 0, \quad \varepsilon > 0.$$

A set  $C$  is said to satisfy *condition A* if

$$(2.4) \quad \sup_{\substack{x_1, x_2 \in C \\ x_1 \neq x_2}} \frac{|\|x_1\| - \|x_2\||}{\|x_1 - x_2\|^2} = M < \infty.$$

The isonormal process  $L$  restricted to such set  $C$ , satisfies the following bound, which is derived in Samorodnitsky (1986).

$$(2.5) \quad P\left\{\sup_{x \in C} Lx > \lambda_0 \sigma + \sum_{j=1}^{\infty} \varepsilon_j \lambda_j\right\} \leq N_C(\varepsilon_1) \cdot \frac{1}{\sqrt{2\pi}} \lambda_0^{-1} e^{-\frac{\lambda_0^2}{2}} \\ + \frac{1}{\sqrt{2\pi}} \lambda_0^{-1} e^{-\frac{\lambda_0^2}{2}} \cdot \sum_{k=2}^{\infty} N_C(\varepsilon_k) \exp\left\{-\frac{1}{2}(\lambda_{k-1} - \rho_k^* \lambda_0)^2\right\}$$

for any positive sequences  $\{\varepsilon_j\}_{j=1}^{\infty}$ ,  $\{\lambda_j\}_{j=0}^{\infty}$ ,  $\varepsilon_j \rightarrow 0$  as  $j \rightarrow \infty$ , satisfying the following conditions for all  $k \geq 2$ .

$$(2.6) \quad \lambda_{k-1} \geq 1,$$

$$(2.7) \quad \rho_k^* < \frac{\lambda_{k-1}}{\lambda_0} < \frac{1}{\rho_k^*}.$$

Here

$$(2.8) \quad \rho_k^* := \frac{2M\sigma + 1}{2\sigma} \varepsilon_{k-1}$$

and

$$\sigma := \sup_{x \in C} \|x\|, \quad \underline{\sigma} := \inf_{x \in C} \|x\| > 0$$

The proof of Theorem 2.1 is based on the bound (2.5).

Remark 2.1. The following property of metric entropy is used in the sequel.

For any  $C_1 \subset C_2$ , any  $\varepsilon > 0$

$$(2.9) \quad N_{C_1}(\varepsilon) \leq N_{C_2}\left(\frac{\varepsilon}{2}\right).$$

This relation is semi-evident; details can be found in Samorodnitsky (1986).

THEOREM 2.1.

Suppose there exists a function  $H(s, t)$  such that the following two conditions hold

$$(2.10) \quad \ell n N_{x_0}(\delta_1, \delta_2, \varepsilon) \leq H(\delta_1, \varepsilon)$$

for any  $\varepsilon > 0$ , any  $0 < \delta_1 < \delta_2 \leq \theta$ ,  $\theta$  is a fixed constant.

$$(2.11) \quad \lim_{s \rightarrow 0} \int_0^s H(s, t)^{1/2} dt = 0.$$

Then the isonormal process  $L$  restricted to  $C$  is continuous at  $x_0$ .

Remark 2.2. An alternative sufficient condition for continuity of the isonormal process at  $x_0$  that follows directly from (1.1) is

$$(2.12) \quad \int_0^1 H_{C_{x_0}}(\delta)(t)^{1/2} dt < \infty \quad \text{for all } \delta > 0 \text{ small enough.}$$

It is easily seen that the condition given by Theorem 2.1 is strictly weaker than (2.12). (Suppose that (2.12) holds for all  $\delta \leq \theta$ . Then take

$H(s, t) := \ell n N_{x_0}(s, \theta, \frac{t}{2})$ . Then (2.10) holds via (2.9), while (2.12) and (2.9)

together imply (2.11). The examples given in the end of this section represent situations in which Theorem 2.1 works while (2.12) fails.)

Proof of the theorem.

Note that  $\tilde{H}(s, t) := \ell n N_{x_0}(s, \theta, \frac{t}{2})$  also possesses properties (2.10) and (2.11). Consequently we may and will assume that  $H(s, t)$  is nonincreasing in  $t$  for each  $s$ . Fix  $0 < u < 1$  and define

$$(2.13) \quad H_u(s, t) := H(s, \frac{u}{2}t)$$

$$(2.14) \quad H_1(s, t)^{1/2} := H_u(s, t)^{1/2} + |\ell n s| + \ell n |\ell n t|$$

$$(2.15) \quad H_2(s, t)^{1/2} := H_1(s, t)^{1/2} + \frac{1}{s} \left[ \int_0^s H_1(s, u)^{1/2} du \right]^{1/2}.$$

These functions still possess properties (2.11) and (at least for small  $t$ )

(2.10). Set

$$(2.16) \quad I(s) := \int_0^s H_2(s, t)^{1/2} dt$$

and

$$(2.17) \quad g(s) := \left[ \sup_{u \leq s} \{I(u) + I(u)^{1/2}\} \right]^{1/2}.$$

Then  $g(s) \downarrow 0$  as  $s \rightarrow 0$ . We are going to show that for an appropriate version of the process

$$(2.18) \quad \limsup_{\delta \rightarrow 0} \left\{ \frac{|Lx - Lx_0|}{g(\|x - x_0\|)} : x \in C_{x_0}(\delta) \right\} \leq 1 \quad \text{a.s.}$$

This implies, of course, that the isonormal process  $L$  restricted to  $C$  is continuous at  $x_0$ . As the supremum in (2.18) is taken over a decreasing family of sets, it is enough to prove that for any  $\alpha > 0$

$$(2.19) \quad \lim_{\delta \rightarrow 0} P \left[ \sup \left\{ \frac{|Lx - Lx_0|}{g(\|x - x_0\|)} : x \in C_{x_0}(\delta) \right\} > 1 + \alpha \right] = 0.$$

Set  $\delta_i := \delta u^i$ ,  $i = 0, 1, 2, \dots$ . Then  $\delta_i \downarrow 0$  as  $i \rightarrow \infty$ . Consequently the monotonicity of  $g(s)$  implies that

$$\begin{aligned}
(2.20) \quad & P\left\{\sup\left\{\frac{|Lx - Lx_0|}{g(\|x - x_0\|)} : x \in C_{x_0}(\delta)\right\} > 1 + \alpha\right\} \\
& \leq \sum_{i=0}^{\infty} P\left\{\sup\left\{\frac{|Lx - Lx_0|}{g(\|x - x_0\|)} : x \in C_{x_0}(\delta_i) \cap \overline{C_{x_0}(\delta_{i+1})}\right\} > 1 + \alpha\right\} \\
& \leq \sum_{i=0}^{\infty} P\left\{\sup\{|Lx - Lx_0| : x \in C_{x_0}(\delta_i) \cap \overline{C_{x_0}(\delta_{i+1})}\} > (1 + \alpha)g(\delta_{i+1})\right\}.
\end{aligned}$$

We are going to estimate the probabilities in the last sum in (2.20). Denote

$$C_i := \{x - x_0, x \in C_{x_0}(\delta_i) \cap \overline{C_{x_0}(\delta_{i+1})}\}.$$

We obtain by the linearity of the isonormal process that

$$\begin{aligned}
(2.21) \quad & P\left\{\sup\{|Lx - Lx_0| : x \in C_{x_0}(\delta_i) \cap \overline{C_{x_0}(\delta_{i+1})}\} > (1 + \alpha)g(\delta_{i+1})\right\} \\
& = P\left\{\sup_{x \in C_i} |Lx| > (1 + \alpha)g(\delta_{i+1})\right\}.
\end{aligned}$$

We would like to apply the bound (2.5) to the last probability on (2.21).

However,  $C_i$  may not satisfy condition A, thus we first define

$$\hat{C}_i := \left\{x \cdot \frac{\delta_i}{\|x\|}, x \in C_i\right\}.$$

Clearly,

$$\sup_{x \in \hat{C}_i} |Lx| = \sup_{x \in C_i} \left\{|Lx| \cdot \frac{\delta_i}{\|x\|}\right\} \geq \sup_{x \in C_i} |Lx|.$$

Thus

$$(2.22) \quad P\left\{\sup_{x \in \hat{C}_i} |Lx| > (1 + \alpha)g(\delta_{i+1})\right\} \leq P\left\{\sup_{x \in C_i} |Lx| > (1 + \alpha)g(\delta_{i+1})\right\}$$

Moreover, the points in  $\hat{C}_i$  have equal norms  $(\delta_i)$ , and so  $\hat{C}_i$  satisfies condition A with  $M=0$ . We apply the bound (2.5) to  $\hat{C}_i$ , taking  $\sigma = \underline{\sigma} = \delta_i$ ,  $M=0$ . For any two sequences  $\{\varepsilon_j^{(i)}\}_{j=1}^{\infty}$ ,  $\{\lambda_j^{(i)}\}_{j=0}^{\infty}$  satisfying (2.6) and (2.7) we thus obtain

$$(2.23) \quad P\{\sup_{x \in \hat{C}_i} |Lx| > \lambda_0^{(i)} \delta_i + \sum_{j=1}^{\infty} \varepsilon_j^{(i)} \lambda_j^{(i)}\} \leq \sqrt{\frac{2}{\pi}} (\lambda_0^{(i)})^{-1} \exp\{-\frac{1}{2}(\lambda_0^{(i)})^2\} N_{\hat{C}_i}(\varepsilon_1^{(i)}) \\ + \sqrt{\frac{2}{\pi}} (\lambda_0^{(i)})^{-1} \exp\{-\frac{1}{2}(\lambda_0^{(i)})^2\} \sum_{j=2}^{\infty} N_{\hat{C}_i}(\varepsilon_j^{(i)}) \exp\{-\frac{1}{2}(\lambda_{j-1}^{(i)} - \rho_j^* \lambda_0^{(i)})^2\}.$$

Note that for any  $x_1, x_2 \in C_i$

$$\|x_1 \cdot \frac{\delta_i}{\|x_1\|} - x_2 \cdot \frac{\delta_i}{\|x_2\|}\| = \delta_i \|\frac{x_1}{\|x_1\|} - \frac{x_2}{\|x_1\|} + \frac{x_2}{\|x_1\|} - \frac{x_2}{\|x_2\|}\| \\ \leq \delta_i (\frac{1}{\|x_1\|} \|x_1 - x_2\| + \|x_2\| \cdot |\frac{1}{\|x_1\|} - \frac{1}{\|x_2\|}|) \\ \leq \frac{2\delta_i}{\|x_1\|} \cdot \|x_1 - x_2\| \leq \frac{2\delta_i}{\delta_{i+1}} \|x_1 - x_2\| \\ = \frac{2}{u} \|x_1 - x_2\|.$$

Consequently for any  $\varepsilon > 0$

$$(2.24) \quad N_{\hat{C}_i}(\varepsilon) \leq N_{\hat{C}_i}(\varepsilon \cdot \frac{u}{2}) = N(C_{x_0}(\delta_i) \cap \overline{C_{x_0}(\delta_{i+1})}, \varepsilon \cdot \frac{u}{2})$$

Thus, (2.23) can be rewritten as

$$(2.25) \quad P\{\sup_{x \in \hat{C}_i} |Lx| > \lambda^{(i)} \delta_i + \sum_{j=1}^{\infty} \varepsilon_j^{(i)} \lambda_j^{(i)}\} \\ \leq \sqrt{\frac{2}{\pi}} \lambda^{(i)}^{-1} \exp\{-\frac{1}{2} \lambda^{(i)2}\} N_{x_0}(\delta_{i+1}, \delta_i, \frac{\varepsilon_1^{(i)} u}{2}) \\ + \sqrt{\frac{2}{\pi}} \lambda^{(i)}^{-1} \exp\{-\frac{1}{2} \lambda^{(i)2}\} \sum_{j=2}^{\infty} N_{x_0}(\delta_{i+1}, \delta_i, \frac{\varepsilon_j^{(i)} u}{2}) \exp\{-\frac{1}{2}(\lambda_{j-1}^{(i)} - \rho_j^* \lambda^{(i)})^2\},$$

where  $\lambda(i)$  denotes  $\lambda_0^{(i)}$ . Now we specify the  $\varepsilon_j^{(i)}$  and  $\lambda_j^{(i)}$  sequences. Set

$$(2.26) \quad \lambda(i) = \lambda_0^{(i)} := \frac{g(\delta_{i+1})}{\delta_i},$$

$$(2.27) \quad \varepsilon_j^{(i)} := B_1 \delta_i 2^{-j}, \quad j = 1, 2, \dots,$$

$$(2.28) \quad \lambda_j^{(i)} := B_2 \lambda(i) \frac{H_2(\delta_{i+1}, 2^{-j} \delta_{i+1})^{1/2}}{H_2(\delta_{i+1}, 2^{-1} \delta_{i+1})^{1/2}}, \quad j = 1, 2, \dots,$$

for some positive constants  $B_1, B_2$  to be specified later in order to meet certain conditions. Note that  $g(s) \geq s^{1/2} |\ln s|^{1/2}$ , consequently, at least for small values of  $\delta$ ,  $\lambda_j^{(i)} \geq 1$  for all  $i$  and  $j$ . Furthermore, the condition

$\rho_j^* < \frac{\lambda_j^{(i)}}{\lambda(i)}$  holds for all  $i \geq 0, j \geq 2$  if and only if

$$(2.29) \quad B_2 > \frac{B_1}{4}.$$

The second part of condition (2.7), namely  $\frac{\lambda_j^{(i)}}{\lambda(i)} < \frac{1}{\rho_j^*}$ , becomes, after a substitution,

$$(2.30) \quad \frac{H_2(\delta_{i+1}, 2^{-1} \delta_{i+1})^{1/2}}{B_2 H_2(\delta_{i+1}, 2^{-(j-1)} \delta_{i+1})^{1/2}} > B_1 2^{-j}.$$

Note that for any  $0 < d < 1$  we have

$$(2.31) \quad \frac{H_2(t, dt)^{1/2}}{H_2(t, t)^{1/2}} = \frac{H_1(t, dt)^{1/2} + \frac{1}{t} \left[ \int_0^t H_1(t, u)^{1/2} du \right]^{1/2}}{H_1(t, t)^{1/2} + \frac{1}{t} \left[ \int_0^t H_1(t, u)^{1/2} du \right]^{1/2}} \\ \leq \frac{\frac{1}{dt} \int_0^t H_1(t, u)^{1/2} du + \frac{1}{t} \left[ \int_0^t H_1(t, u)^{1/2} du \right]^{1/2}}{\frac{1}{t} \left[ \int_0^t H_1(t, u)^{1/2} du \right]^{1/2}} \\ \leq \frac{1}{d} \left[ \int_0^t H_1(t, u)^{1/2} du \right]^{1/2} + 1 \leq \frac{K}{d},$$

where  $K < \infty$  is a constant that does not depend on  $t$  and  $d$ . Thus, the following condition

$$\frac{2^{-(j-2)}}{B_2 K} > B_1 2^{-j},$$

which is equivalent to

$$(2.32) \quad B_1 B_2 < \frac{4}{K},$$

implies (2.30) for all  $i \geq 0$ ,  $j \geq 2$ . Consequently, if we choose  $B_1$  and  $B_2$  to satisfy the restrictions (2.29) and (2.32), the condition (2.7) will be satisfied. Furthermore,

$$\begin{aligned} \lambda(i) \delta_i + \sum_{j=1}^{\infty} \varepsilon_j^{(i)} \lambda_j^{(i)} &= \lambda(i) \delta_i \left[ 1 + \frac{B_1 B_2}{H_2(\delta_{i+1}, 2^{-1} \delta_{i+1})^{1/2}} \sum_{j=1}^{\infty} 2^{-j} H_2(\delta_{i+1}, 2^{-j} \delta_{i+1})^{1/2} \right] \\ &\leq \lambda(i) \delta_i \left[ 1 + 2B_1 B_2 \frac{\int_0^{2^{-1} \delta_{i+1}} H_2(\delta_{i+1}, u)^{1/2} du}{\delta_{i+1} H_2(\delta_{i+1}, 2^{-1} \delta_{i+1})^{1/2}} \right]. \end{aligned}$$

Here the monotonicity of  $H_2(s, t)$  in  $t$  has been used. An argument similar to that in (2.31) yields that for  $0 < d < 1$

$$(2.33) \quad \frac{\int_0^t H_2(t, u)^{1/2} du}{dt H_2(t, dt)^{1/2}} \leq \frac{K}{d}$$

where  $K$  is the same constant as in (2.31), independent of  $t$  and  $d$ . Consequently

$$\lambda(i) \delta_i + \sum_{j=1}^{\infty} \varepsilon_j^{(i)} \lambda_j^{(i)} \leq \lambda(i) \delta_i (1 + 2B_1 B_2 K).$$

If we also choose  $B_1$  and  $B_2$  in such a way that

$$(2.34) \quad 2B_1 B_2 K \leq \alpha$$

then we get

$$\lambda(i)\delta_i + \sum_{j=1}^{\infty} \varepsilon_j^{(i)} \lambda_j^{(i)} \leq (1+\alpha)\lambda(i)\delta_i = (1+\alpha)g(\delta_{i+1}).$$

Consequently, (2.25) implies that

$$(2.35) \quad P\left\{\sup_{x \in C_i} |Lx| > (1+\alpha)g(\delta_{i+1})\right\} \leq \sqrt{\frac{2}{\pi}} \lambda(i)^{-1} \exp\left\{-\frac{1}{2}\lambda(i)^2\right\} N_{x_0}(\delta_{i+1}, \delta_i, \frac{\varepsilon_1^{(i)} u}{2}) \\ + \sqrt{\frac{2}{\pi}} \lambda(i)^{-1} \exp\left\{-\frac{1}{2}\lambda(i)^2\right\} \sum_{j=2}^{\infty} N_{x_0}(\delta_{i+1}, \delta_i, \frac{\varepsilon_j^{(i)} u}{2}) \exp\left\{-\frac{1}{2}(\lambda_{j-1}^{(i)} - \rho_j^* \lambda(i))^2\right\}.$$

For every fixed  $i=0,1,2,\dots$  let  $a_i$  and  $b_i$  be the first and the second terms, respectively, in the right hand side of (2.35). Then (2.20), (2.21), (2.22) and (2.35) imply that

$$(2.36) \quad P\left\{\sup\left\{\frac{|Lx - Lx_0|}{g(\|x - x_0\|)} : x \in C_{x_0}(\delta)\right\} > 1+\alpha\right\} \leq \sum_{i=0}^{\infty} a_i + \sum_{i=0}^{\infty} b_i.$$

We are going to estimate each of these sums separately. Letting  $c$  be a finite positive constant that may vary from line to line we obtain

$$\sum_{i=0}^{\infty} a_i \leq c \sum_{i=0}^{\infty} \exp\left\{-\frac{g(\delta_{i+1})^2}{2\delta_i^2} + \ell n N_{x_0}(\delta_{i+1}, \delta_i, \frac{\varepsilon_1^{(i)} u}{2})\right\} \\ \leq c \sum_{i=0}^{\infty} \exp\left\{-\frac{g(\delta_{i+1})^2}{2\delta_i^2} + H(\delta_{i+1}, \frac{\varepsilon_1^{(i)} u}{2})\right\} \\ \leq c \sum_{i=0}^{\infty} \exp\left\{-\frac{g(\delta_{i+1})^2}{2\delta_i^2} + H_2(\delta_{i+1}, \varepsilon_1^{(i)})\right\} \\ \leq c \sum_{i=0}^{\infty} \exp\left\{-\frac{1}{2} \frac{I(\delta_{i+1})}{\delta_i^2} + H_2(\delta_{i+1}, B_1 2^{-1} \delta_i)\right\}$$



$$\begin{aligned}
&\leq c \sum_{i=0}^{\infty} \exp\left\{-\frac{u}{2} \frac{H_2(\delta_{i+1}, \delta_{i+1})^{1/2}}{\delta_i} + H_2(\delta_{i+1}, B_1 2^{-1} \delta_{i+1})\right\} \\
&= c \sum_{i=0}^{\infty} \exp\left\{-\frac{u}{2} \frac{H_2(\delta_{i+1}, \delta_{i+1})^{1/2}}{\delta_i} \left[1 - \frac{2\delta_i}{u} \cdot \frac{H_2(\delta_{i+1}, B_1 2^{-1} \delta_{i+1})}{H_2(\delta_{i+1}, \delta_{i+1})^{1/2}}\right]\right\}
\end{aligned}$$

for  $\delta$  small enough. Note that (2.31) implies that

$$\frac{H_2(\delta_{i+1}, B_1 2^{-1} \delta_{i+1})^{1/2}}{H_2(\delta_{i+1}, \delta_{i+1})^{1/2}} \leq \max\{1, \frac{2K}{B_1}\}$$

while

$$\delta_i H_2(\delta_{i+1}, B_1 2^{-1} \delta_{i+1})^{1/2} \leq u^{-1} \max(1, \frac{2K}{B_1}) \delta_{i+1} H_2(\delta_{i+1}, \delta_{i+1})^{1/2}$$

and the last expression converges uniformly in  $i$  to zero as  $\delta \rightarrow 0$ . Consequently, for  $\delta$  small

$$\begin{aligned}
\sum_{i=0}^{\infty} a_i &\leq c \sum_{i=0}^{\infty} \exp\left\{-\frac{u}{4\delta_i} H_2(\delta_{i+1}, \delta_{i+1})^{1/2}\right\} \leq c \sum_{i=0}^{\infty} \exp\{-|\ell n \delta_{i+1}|\} = c \sum_{i=0}^{\infty} \delta_{i+1} \\
&= \frac{c\delta u}{1-u} \rightarrow 0
\end{aligned}$$

as  $\delta \rightarrow 0$ . Next we consider the sum  $\sum_{i=0}^{\infty} b_i$ . Note that for all  $j \geq 2$

$$\begin{aligned}
(2.34) \quad \lambda_{j-1}^{(1)} - \rho_j^{*\lambda}(1) &= \lambda(1) [B_2 \frac{H_2(\delta_{i+1}, 2^{-(j-1)} \delta_{i+1})^{1/2}}{H_2(\delta_{i+1}, 2^{-1} \delta_{i+1})^{1/2}} - B_1 2^{-j}] \\
&\geq B_2 \lambda(1) \frac{H_2(\delta_{i+1}, 2^{-(j-1)} \delta_{i+1})^{1/2}}{H_2(\delta_{i+1}, 2^{-1} \delta_{i+1})^{1/2}} (1 - \frac{B_1}{4B_2}) \\
&\geq \frac{1}{2} B_2 \lambda(1) \frac{H_2(\delta_{i+1}, 2^{-(j-1)} \delta_{i+1})^{1/2}}{H_2(\delta_{i+1}, 2^{-1} \delta_{i+1})^{1/2}},
\end{aligned}$$

whenever  $B_1$  and  $B_2$  are chosen to satisfy

$$(2.38) \quad \frac{B_2}{B_1} \geq \frac{1}{2}.$$

Note that (2.38) implies (2.29). Using (2.37) we get that for small  $\delta$

$$\begin{aligned} & \sum_{j=2}^{\infty} N_{x_0}(\delta_{i+1}, \delta_i, \frac{\varepsilon_j^{(i)} u}{2}) \exp\{-\frac{1}{2}(\lambda_{j-1}^{(i)} - \rho_j^* \lambda^{(i)})^2\} \\ & \leq \sum_{j=2}^{\infty} \exp\{+H_2(\delta_{i+1}, B_1 \delta_i 2^{-j}) - \frac{B_2^2}{8} \frac{g(\delta_{i+1})^2}{\delta_i^2} \cdot \frac{H_2(\delta_{i+1}, 2^{-(j-1)} \delta_{i+1})}{H_2(\delta_{i+1}, 2^{-1} \delta_{i+1})}\} \\ & \leq \sum_{j=2}^{\infty} \exp\{-\frac{B_2^2}{8\delta_i^2} \cdot \frac{H_2(\delta_{i+1}, 2^{-(j-1)} \delta_{i+1})}{H_2(\delta_{i+1}, 2^{-1} \delta_{i+1})} \cdot \frac{I(\delta_{i+1})}{\delta_i}\} \\ & \quad \times [1 - \frac{8}{B_2^2} \max(1, \frac{4K_u^2}{B_1^2}) \frac{H_2(\delta_{i+1}, 2^{-1} \delta_{i+1}) \delta_i^2}{I(\delta_{i+1})}] \\ & \leq \sum_{j=2}^{\infty} \exp\{-\frac{B_2^2 u}{8\delta_i^2} \cdot \frac{H_2(\delta_{i+1}, 2^{-(j-1)} \delta_{i+1})}{H_2(\delta_{i+1}, 2^{-1} \delta_{i+1})} \cdot H_2(\delta_{i+1}, \delta_{i+1})^{1/2}\} \\ & \quad \times [1 - \frac{8}{B_2^2 u} \cdot \max(1, \frac{4K_u^2}{B_1^2}) \frac{H_2(\delta_{i+1}, 2^{-1} \delta_{i+1})}{H_2(\delta_{i+1}, \delta_{i+1})^{1/2}} \cdot \delta_i] \\ & \leq \sum_{j=2}^{\infty} \exp\{-\frac{B_2^2 u}{8\delta_i^2} \cdot \frac{H_2(\delta_{i+1}, 2^{-(j-1)} \delta_{i+1})^{1/2}}{H_2(\delta_{i+1}, 2^{-1} \delta_{i+1})^{1/2}} \cdot H_2(\delta_{i+1}, \delta_{i+1})^{1/2}\} \\ & \quad \times [1 - \frac{8}{B_2^2 u} \cdot \max(1, \frac{4K_u^2}{B_1^2}) \frac{H_2(\delta_{i+1}, 2^{-1} \delta_{i+1})^{1/2}}{H_2(\delta_{i+1}, \delta_{i+1})^{1/2}} \cdot \delta_i H_2(\delta_{i+1}, 2^{-1} \delta_{i+1})^{1/2}] \end{aligned}$$

$$\leq \sum_{j=2}^{\infty} \exp\left\{-\frac{B_2^2 u}{8} \cdot \frac{1}{2K} \cdot \frac{H_2(\delta_{i+1}, 2^{-(j-1)}\delta_{i+1})^{1/2}}{\delta_i}\right\}$$

$$\times [1 - \frac{8}{B_2^2 u} \cdot \max(1, \frac{4K^2 u^2}{B_1^2}) \cdot (2K)^2 \cdot \delta_i H_2(\delta_{i+1}, \delta_{i+1})^{1/2}].$$

But  $\delta H_2(\delta, \delta)^{1/2} \rightarrow 0$  as  $\delta \rightarrow 0$ . Consequently for small  $\delta$ , uniformly over  $i$

$$\sum_{j=2}^{\infty} N_{x_0}(\delta_{i+1}, \delta_i, \frac{\epsilon_j^{(1)} u}{2}) \exp\left\{-\frac{1}{2}(\lambda_{j-1}^{(1)} - \rho_j^* \lambda^{(1)})^2\right\}$$

$$\leq \sum_{j=2}^{\infty} \exp\left\{-\frac{B_2^2 u}{32K} \cdot \frac{H_2(\delta_{i+1}, 2^{-(j-1)}\delta_{i+1})^{1/2}}{\delta_i}\right\}$$

$$\leq \sum_{j=2}^{\infty} \exp\{-2\ell n |\ell n 2^{-(j-1)}\delta_{i+1}|\}$$

$$\leq \sum_{j=2}^{\infty} ((j-1)\ell n 2)^{-2} < \infty$$

independently of  $i$ . Consequently, for small  $\delta$

$$\sum_{i=0}^{\infty} b_i \leq c \sum_{i=0}^{\infty} a_i.$$

Thus  $\sum_{i=0}^{\infty} b_i$  is finite for small  $\delta$ , and  $\sum_{i=0}^{\infty} b_i \rightarrow 0$  as  $\delta \rightarrow 0$ . Consequently, (2.19)

is proven and then (2.18) follows. To finish the proof we have to show that it is possible to choose  $B_1$  and  $B_2$  to satisfy conditions (2.32), (2.34), (2.38).

This is simple. For any  $B_2 > 0$  set  $B_1(B_2) = 2B_2$ . This satisfies (2.38) independently of  $B_2$ . Then  $B_2 \cdot B_1(B_2) = 2B_2^2$ . Consequently taking  $B_2$  small enough we get both (2.32) and (2.34) satisfied. This completes the proof.

The following two examples are taken from Dudley (1973).

**Example 2.1.** Let  $\{a_k\}$  be a sequence of positive numbers with  $a_k \downarrow 0$ . For  $k=1,2,\dots$ , let  $C_k$  be a cube of dimension  $k^2$  and side  $2a_k/k^2$  centered at 0. Let the cubes  $C_k$  lie in orthogonal subspaces. Let  $C = \bigcup_{k=1}^{\infty} C_k$ .

Consider the origin  $x_0 = 0$ . Letting  $a_k \downarrow 0$  slowly, we can make  $\varepsilon^2 H_{C_0(\delta)}(\varepsilon) \rightarrow 0$  as slowly as desired, thus the integral (2.12) can diverge for all  $\delta > 0$ . Nevertheless, the isonormal process is continuous a.s. at  $x_0 = 0$ . We show that Theorem 2.1 works here.

For every fixed  $k$  and  $\varepsilon$

$$N(C_k, \varepsilon) \leq \max\left\{\left(\frac{2a_k}{k\varepsilon}\right)k^2, 1\right\}$$

and for fixed  $0 < \delta_1 < \delta_2$ ,  $\varepsilon > 0$

$$(2.39) \quad N_0(\delta_1, \delta_2, \varepsilon) \leq N(\overline{C_0(\delta_1)}, \frac{\varepsilon}{2}) \leq \sum_{k=1}^{n(\delta_1)} \left(\frac{4a_k}{k\varepsilon}\right)k^2 + 1,$$

where

$$n(s) := \max\{k : \frac{a_k}{k} \geq s\}.$$

Define also

$$m(s) := \max\{k : \frac{a_k}{k} \geq s^{1/4}\}$$

Both  $n(s)$  and  $m(s)$  increase to infinity as  $s \downarrow 0$ . Define

$$(2.40) \quad H(s, t) := \ln\left[\sum_{k=1}^{n(s)} \left(\frac{4a_k}{kt}\right)k^2 + 1\right].$$

We have to show that

$$(2.41) \quad \lim_{s \rightarrow 0} \int_0^s \left\{ \ln\left[\sum_{k=1}^{n(t)} \left(\frac{4a_k}{kt}\right)k^2\right] \right\}^{1/2} dt = 0.$$

The obvious relations  $\ln(x+y) \leq \ln 2x + \ln 2y$ ,  $x, y \geq 1$  and  $(x+y)^{1/2} \leq x^{1/2} + y^{1/2}$  yield that

$$\begin{aligned}
 (2.42) \quad & \int_0^s \{ \ell n [ \sum_{k=1}^{n(s)} (\frac{4a_k}{kt})^2 ] \}^{1/2} dt \\
 & \leq \int_0^s \{ \ell n [ \sum_{k=1}^{m(s)} (\frac{4a_k}{kt})^2 ] \}^{1/2} dt + \int_0^s \{ \ell n [ \sum_{k=m(s)+1}^{n(s)} (\frac{4a_k}{kt})^2 ] \}^{1/2} dt + s(2\ell n 2)^{1/2}.
 \end{aligned}$$

Denote by  $I_1$  and  $I_2$  the first and second integrals in the right hand side of (2.42). We are going to show that

$$\lim_{s \rightarrow 0} I_1 = \lim_{s \rightarrow 0} I_2 = 0.$$

We have, for small  $s$ ,

$$\begin{aligned}
 I_1 & \leq \int_0^s \{ \sum_{k=1}^{m(s)} [k^2 \ell n(\frac{4a_k}{kt}) + \ell n 2] \}^{1/2} dt \\
 & \leq s(m(s)\ell n 2)^{1/2} + \int_0^s \sum_{k=1}^{m(s)} k [\ell n(\frac{4a_k}{kt})]^{1/2} dt \\
 & \leq s(m(s)\ell n 2)^{1/2} + \sum_{k=1}^{m(s)} \int_0^s k [\ell n(\frac{4a_1}{t})]^{1/2} dt \\
 & \leq s(m(s)\ell n 2)^{1/2} + \frac{m(s)(m(s)+1)}{2} \cdot 2s [\ell n(\frac{4a_1}{s})]^{1/2}
 \end{aligned}$$

and this goes to zero as  $s \rightarrow 0$  because  $m(s) \leq a_1 s^{-1/4}$ . Further, for every  $m(s) < k \leq n(s)$

$$(2.43) \quad (\frac{4a_k}{kt})^2 = [(\frac{4a_k}{kt})^{\frac{kt}{4a_k}}]^{\frac{4ka_k}{t}} \leq (\frac{1}{e})^{\frac{4ka_k}{t}} \leq (\frac{1}{e})^{\frac{4a_k^2}{st}} \leq \exp\{\frac{4a_k^2}{e \cdot s \cdot t}\},$$

where the first inequality in (2.43) follows from the fact that

$$\max_{x>0} x^{\frac{1}{x}} = e^{\frac{1}{e}}$$

and the second inequality follows from the fact that  $a_k/k \geq s$ . We conclude that

$$\begin{aligned}
(2.44) \quad I_2 &\leq \int_0^s \{ \ln[n(s) \exp(\frac{4a_m^2(s)+1}{e \cdot s \cdot t})] \}^{1/2} dt \\
&\leq s[\ln n(s)]^{1/2} + 2e^{-1/2} a_{m(s)+1} \cdot s^{-1/2} \cdot \int_0^s t^{-1/2} \cdot dt \\
&= s[\ln n(s)]^{1/2} + 4e^{-1/2} \cdot a_{m(s)+1}.
\end{aligned}$$

The first term in (2.44) converges to zero because  $n(s) \leq s^{-1} a_{n(s)}$ , while the second term converges to zero because  $m(s) \rightarrow \infty$  as  $s \rightarrow 0$ . Therefore, the conditions of Theorem 2.1 hold.

Remark 2.3. Note that the GC property of  $C$  follows. We have just proved that the isonormal process is continuous at  $x_0 = 0$ . Certainly, for any  $x_0 \in C$  other than the origin, for any sufficiently small  $\delta > 0$  the metric entropy of  $C_{x_0}(\delta)$  is bounded by a logarithmic function of  $\epsilon$ , and so the integral (2.12) is finite. In this example, consequently, we have proved that  $C$  is a GC class by proving that the isonormal process is continuous at the (only) "difficult" point  $x_0 = 0$ .

Example 2.2. Again, let  $\{a_k\}$  be a sequence of positive numbers with  $a_k \rightarrow 0$ . Let

$$C = \{\phi_n \cdot a_n (\ln n)^{-1/2}, n \geq 2\} \cup \{0\},$$

$\phi_n$  orthonormal. Consider  $x_0 = 0$ . If  $a_k \rightarrow 0$  slowly ( $a_k = (\ln \ln k)^{-1/2}$  is slow enough) then the integral (2.12) diverges for all  $\delta > 0$ . Let us show that Theorem 2.1 can be applied to this example as well. Set

$$M(s) := \min\{n : \frac{a_n^2}{\ln n} \leq s^2\}.$$

Then, for any  $0 < \delta_1 < \delta_2$ ,  $\epsilon > 0$

$$(2.45) \quad N_0(\delta_1, \delta_2, \epsilon) = [\min(M(\epsilon), M(\delta_1)) - M(\delta_2)]_+ + 1.$$

This implies that

$$(2.46) \quad N_0(\delta_1, \delta_2, \epsilon) \leq 2M(\delta_1).$$

Set

$$(2.47) \quad H(s, t) := \ell_n 2 + \ell_n M(s).$$

Then

$$\int_0^s H(s, t)^{1/2} dt = s[\ell_n 2 + \ell_n M(s)]^{1/2}$$

and we have to show that

$$\lim_{s \rightarrow 0} s^2 \ell_n M(s) = 0.$$

But this is clear, since

$$\frac{s^2 a_{M(s)-1}}{\ell_n(M(s)-1)} > s^2,$$

so

$$M(s) < 2 \exp\{s^{-2} \cdot a_{M(s)-1}\},$$

and also  $M(s) \uparrow \infty$  as  $s \downarrow 0$ .

Remark 2.4. Again, we have proved the GC property of  $\mathcal{C}$ , since at any point  $x_0 \in \mathcal{C}$  other than the origin the isonormal process cannot have a discontinuity.

### Appendix

Theorem 1.1 is proven here. The idea of the proof is the same as that of Marcus and Shepp (1971) in the case of Gaussian processes on  $[0,1]$ .

Let  $C^*$  be a countable dense subset of  $C$ . For any  $I \subset C$ , any  $\varepsilon > 0$ , define

$$A_\varepsilon(I) = \left\{ \omega \left| \begin{array}{l} \text{there is an } r > 0 \text{ such that for every } \delta > 0 \text{ there} \\ \text{are } t_1 \in C^* \cap I \text{ and } t_2 \in C^* \text{ such that } d(t_1, t_2) < \delta \\ \text{and } |X(t_1) - X(t_2)| \geq \varepsilon + r. \end{array} \right. \right\}$$

Lemma A.1.

$$P(A_\varepsilon(I)) = 0 \text{ or } 1.$$

Proof of the lemma.

Let  $s_1, s_2, \dots$  be a numeration of the points of  $C^*$ . Then there exists a sequence of orthonormal Gaussian variables  $Y_1, Y_2, \dots$  and real numbers  $\{a_{ij}\}$ ,  $i \leq j$ , such that for each  $i$

$$(A.1) \quad X(s_i) = \sum_{j=1}^i a_{ij} Y_j.$$

Let  $a_{ij} = 0$  for  $j > i$ , and let for  $t_1, t_2 \in C^*$   $i_1$  and  $i_2$  denote the places of  $t_1$  and  $t_2$  correspondingly in the fixed numeration of  $C^*$ . Then

$$(A.2) \quad \sigma^2(t_1, t_2) = E[X(t_1) - X(t_2)]^2 = \sum_{j=1}^{\infty} (a_{i_1 j} - a_{i_2 j})^2.$$

Note that the covariance function of the process,  $R(s, t)$ , is continuous on  $C \times C$ , since  $X(t)$  is a.s. continuous at each point of  $C$ . Then the function  $\sigma^2(s, t)$  is also continuous on  $C \times C$ , and, because of compactness, it is uniformly continuous. Consequently, for every  $\theta > 0$  there is a  $\eta = \eta(\theta) > 0$  such that  $d(t_1, t_2) \leq \eta$  implies that  $\sigma^2(t_1, t_2) \leq \theta^2$ . If also  $t_1, t_2 \in C^*$ , then (A.2) implies that  $|a_{i_1 j} - a_{i_2 j}| \leq \theta$  for every  $j$ . Let us rewrite the event  $A_\varepsilon(I)$  as



$$A_{\varepsilon}(I) = \left\{ \omega \left| \begin{array}{l} \text{there is an } r > 0 \text{ such that for every } \delta > 0 \text{ there} \\ \text{are } t_1 \in C^* \cap I \text{ and } t_2 \in C^* \text{ such that } d(t_1, t_2) < \delta \\ \text{and } \left| \sum_{j=1}^{\infty} Y_j(\omega)(a_{i_1 j} - a_{i_2 j}) \right| \geq \varepsilon + r. \end{array} \right. \right\}$$

We claim that  $A_{\varepsilon}(I)$  is a tail event for the sequence  $Y_1, Y_2, \dots$ . It is sufficient to show that if  $\omega_1$  and  $\omega_2$  are such that for some finite  $m$   $Y_n(\omega_1) = Y_n(\omega_2)$  for all  $n > m$ , then  $\omega_1 \in A_{\varepsilon}(I)$  implies  $\omega_2 \in A_{\varepsilon}(I)$ .

Suppose the contrary, i.e.  $\omega_1 \in A_{\varepsilon}(I)$  and  $\omega_2 \notin A_{\varepsilon}(I)$ . Then for some positive  $r(\omega_1)$ , for all sufficiently small  $\delta > 0$  there are  $t_1 \in C^* \cap I$  and  $t_2 \in C^*$  such that  $d(t_1, t_2) < \delta$  while

$$\left| \sum_{j=1}^{\infty} Y_j(\omega_1)(a_{i_1 j} - a_{i_2 j}) \right| \geq \varepsilon + r(\omega_1)$$

$$\left| \sum_{j=1}^{\infty} Y_j(\omega_2)(a_{i_1 j} - a_{i_2 j}) \right| < \varepsilon + \frac{1}{2} r(\omega_1).$$

(Here, as before,  $i_1$  and  $i_2$  denote the places of  $t_1$  and  $t_2$  respectively in the fixed numeration of  $C^*$ ). We conclude (recall that all sums have finite numbers of terms) that

$$\begin{aligned} \frac{1}{2} r(\omega_1) &< \left| \sum_{j=1}^{\infty} Y_j(\omega_1)(a_{i_1 j} - a_{i_2 j}) \right| - \left| \sum_{j=1}^{\infty} Y_j(\omega_2)(a_{i_1 j} - a_{i_2 j}) \right| \\ (A.3) \quad &\leq \left| \sum_{j=1}^{\infty} Y_j(\omega_1)(a_{i_1 j} - a_{i_2 j}) - \sum_{j=1}^{\infty} Y_j(\omega_2)(a_{i_1 j} - a_{i_2 j}) \right| \\ &= \left| \sum_{j=1}^m (Y_j(\omega_1) - Y_j(\omega_2))(a_{i_1 j} - a_{i_2 j}) \right|. \end{aligned}$$

The inequality (A.3), however, cannot hold for all positive  $\delta$ , since its left-hand side is positive, while the right-hand side goes to zero as  $\delta \rightarrow 0$ . This contradiction shows that  $A_{\varepsilon}(I)$  is a tail event. Consequently, Kolmogorov's zero-one law implies that  $P(A_{\varepsilon}(I)) = 0$  or  $1$ .

We return now to the proof of Theorem 1.1. For any  $I \subset C$ , any  $\varepsilon > 0$  define

$$B_{\varepsilon}(I) = \left\{ \omega \mid \text{for any } \delta > 0 \text{ there are } t_1 \in C^* \cap I \text{ and } t_2 \in C^* \text{ such that} \right. \\ \left. d(t_1, t_2) < \delta \text{ and } |X(t_1) - X(t_2)| \geq \varepsilon. \right\}$$

Then  $B_{\varepsilon}(I) \subset A_{\varepsilon/2}(I)$ . If  $X(t)$  is not sample continuous then  $P(B_{\varepsilon}(C)) > 0$  for some  $\varepsilon > 0$ . Then  $P(A_{\varepsilon^*}(C)) > 0$  for some  $\varepsilon^* > 0$ . This implies that  $P(A_{\varepsilon^*}(C)) = 1$ .

Let  $d_0 = \sup_{t_1, t_2 \in C} d(t_1, t_2) < \infty$ . The compactness of  $C$  implies that we can cover it by a finite number of compact sets  $C_1^{(1)}, C_2^{(1)}, \dots, C_{k_1}^{(1)}$  of diameter at most  $d_0/2$  each. Since

$$(A.4) \quad A_{\varepsilon^*}(C) = \bigcup_{i=1}^{k_1} A_{\varepsilon^*}(C_i^{(1)})$$

we conclude, that for some  $i_1$ ,  $1 \leq i_1 \leq k_1$ ,  $P(A_{\varepsilon^*}(C_{i_1}^{(1)})) > 0$ . Thus  $P(A_{\varepsilon^*}(C_{i_1}^{(1)})) = 1$ . We divide now  $C_{i_1}^{(1)}$  into compact subsets of diameter at most  $d_0/4$ , and so on. We obtain a sequence of nested compact non-empty sets

$$C = C_1^{(0)} \supset C_{i_1}^{(1)} \supset C_{i_2}^{(2)} \supset \dots$$

with the following two properties: for each  $k=1,2,\dots$   $P(A_{\varepsilon^*}(C_{i_k}^{(k)})) = 1$  and the diameter of  $C_{i_k}^{(k)}$  is at most  $d_0/2^k$ . This sequence has to converge to a point  $t_{\infty} \in C$ . Then, by the definition of  $A_{\varepsilon^*}(I)$ ,

$$P\left(\overline{\lim_{\substack{t \rightarrow t_{\infty} \\ t \in C^*}} X(t) - \lim_{\substack{t \rightarrow t_{\infty} \\ t \in C^*}} X(t) \geq \varepsilon^*}\right) \geq P\left(\bigcap_{k=0}^{\infty} A_{\varepsilon^*}(C_{i_k}^{(k)})\right) = 1.$$

This contradicts the assumption that  $X(t)$  is a.s. continuous at  $t_{\infty}$ . This contradiction shows that  $X(t)$  is sample continuous.

## References

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